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Application of linear hyperbolic PDE to linear quantum fields in curved spacetimes: especially black holes, time machines and a new semi-local vacuum concept

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Abstract

Several situations of physical importance may be modelled by linear quantum fields propagating in fixed spacetime-dependent classical background fields. For example, the quantum Dirac field in a strong and/or time-dependent external electromagnetic field accounts for the creation of electron-positron pairs out of the vacuum. Also, the theory of linear quantum fields propagating on a given background curved spacetime is the appropriate framework for the derivation of black-hole evaporation (Hawking effect) and for studying the question whether or not it is possible in principle to manufacture a time-machine. It is a well-established metatheorem that any question concerning such a linear quantum field may be reduced to a definite question concerning the corresponding classical field theory (i.e. linear hyperbolic PDE with non-constant coefficients describing the background in question) – albeit not necessarily a question which would have arisen naturally in a purely classical context. The focus in this talk will be on the covariant Klein-Gordon equation in a fixed curved background, although we shall draw on analogies with other background field problems and with the time-dependent harmonic oscillator. The aim is to give a sketch-impression of the whole subject of Quantum Field Theory in Curved Spacetime, focussing on work with which the author has been personally involved, and also to mention some ideas and work-in-progress by the author and collaborators towards a new "semi-local" vacuum construction for this subject. A further aim is to introduce, and set into context, some recent advances in our understanding of the general structure of quantum fields in curved spacetimes which rely on classical results from microlocal analysis.

1. Introduction

The subject of Quantum Field Theory in Curved Spacetime has long attracted the interest both of some theoretical physicists and of some workers in the theory of partial differential equations.

To explain what this subject is about, let us start, for example, with the relativistic wave equation (Klein-Gordon equation)

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + m^2 \right) \phi = 0$$

appropriate to the description of a free scalar quantum "particle" of mass m in the flat four-dimensional spacetime of special relativity. Equivalently, in a more compact notation, one writes

$$(\Box + m^2)\phi = 0 \tag{KG0}$$

where $\Box$ stands for $\eta^{ab}\partial_a\partial_b$, and $\eta = \mathrm{diag}(1, -1, -1, -1)$ is the usual Minkowskian metric on $\mathbb{R}^4$. (Above, we have adopted the convention that the speed of light c is equal to 1.)

Bearing in mind that, when we pass from Special to General Relativity, the flat spacetime of Minkowski gets replaced by a curved spacetime $(\mathcal{M}, g)$ – $\mathcal{M}$ a four-dimensional manifold, g a general pseudo-Riemannian metric of signature $(+, -, -, -)$ – it is natural to generalize (1) to the covariant Klein-Gordon equation

$$(\Box_g + m^2)\phi = 0 \tag{KG}$$

where $\Box_g$ now denotes the natural covariant generalization of $\Box$ to this curved spacetime, i.e. the Laplace-Beltrami operator associated with the metric g:

$$\Box_g = g^{ab}\nabla_a\nabla_b$$

$$= |\det g|^{-1/2}\partial_a(|\det g|^{+1/2}g^{ab}\partial_b)$$

$$= g^{ab}\partial_a\partial_b + \text{lower order terms}.$$

(Above, g with indices upstairs denotes the inverse matrix to g with indices downstairs).

KG0 is, of course, the prototype equation for quantum field theory in flat spacetime. It, and its higher-spin counterparts, when coupled together with suitable (nonlinear) interaction terms and when understood, not as classical PDEs but rather in an appropriate quantum-theoretic sense, is believed to describe the behaviour of elementary particles to the extent that the effects of gravity may be ignored. In much the same way, KG, for a given fixed background spacetime $(\mathcal{M}, g)$, and when interpreted in a suitable quantum sense – it will be one of our purposes, below, to assign a definite mathematical meaning to the resulting "quantized KG" – is believed to be a prototype equation for the effect of a given strong external gravitational field on elementary particles propagating in its vicinity. If one's principle interest is in the new features of quantum field theory due to the presence of a strong external gravitational field, it presumably suffices to study this simple linear model. Of course, to study such a theory is a far less ambitious thing to do than to attempt to

formulate a full theory of quantum gravity. In a full theory of quantum gravity, not only would there be interactions between the various matter (i.e. non-gravity) fields but also the gravitational field would itself be dynamical and presumably require a quantum description. But one believes that a theory such as quantized KG should have an interesting domain of validity as an approximate theory, and hopes that by studying it in its domain of validity one might even find some clues as to the nature of quantum gravity itself.

During the last 30 or so years, the resulting subject of Quantum Field Theory in Curved Spacetime and in particular, quantized KG, has turned out to be more rich and interesting than one might have expected. Before turning to the question just what is meant, mathematically, by "quantized KG" let us mention some of the physical predictions resulting from its study. (Note that, of course, physicists did not wait for the mathematical definition of the theory to be completely clear before starting to make their calculations!)

- *The Hawking Effect:* It is the framework within which Hawking made his spectacular 1974 prediction [1] that mini black holes will not in fact be black but rather *hot* with a temperature, given in the case of a spherically symmetric black hole of mass M (and using units where $c = G = k = \hbar = 1$), by the formula $T_{\text{Hawking}} = 1/8\pi M$.

- *The Time-Machine Question:* It also leads to interesting results which suggest that time-travel-to-the-past scenarios, which, worryingly, are seemingly permitted by classical general relativity, are prevented from actually occurring by quantum effects.

At a conceptual level, the attempt to formulate the general theory has raised some important matters of principle, the resolution of which has arguably led to a deeper understanding of quantum field theory in general, even if one is mainly interested in the Minkowski space case. These matters of principle are connected with the following, somewhat paradoxical-seeming circumstance. On the one hand:

- *Particle Creation:* The principal physical phenomenon associated with quantum field theory in curved spacetime is the creation of pairs of particles out of the vacuum.

  Yet, on the other hand,

- *The Problematic Nature of "Particles":* In a general curved spacetime context, the very notion of "particle", becomes vague and ambiguous. Correspondingly, the familiar Minkowski-space notion of a single preferred "vacuum" state has to be abandoned and replaced instead with a preferred family of physically permissible states, amongst which a principle of democracy prevails.

A suitable conceptual-mathematical framework which reflects this state of affairs is now fairly well developed (although there are still also some important gaps as I'll discuss at the end of the talk in Section 6). The key idea is to have a notion

for the field itself which does not rely on any particular concept of "particles" or "vacuum", and to regard this abstract notion of "field" as the fundamental entity. This is achieved by adopting the so called *algebraic approach to quantum field theory* (see [2] and Chapter 3 in [3]) which involves the theory of ∗-algebras and their states and Hilbert-space representations. However, as long as one is concerned with a prototype equation which is *linear* such as (2), the questions which arise concerning the quantum theory ultimately reduce to questions about the underlying classical partial differential equation; the role of the ∗-algebras etc. being to establish a dictionary telling us *which* classical question corresponds to which quantum question. It seems justified, in fact, to say that there is a:

- **Metatheorem**: *Any question concerning the quantum field theory based on KG may be translated into a definite question concerning KG – regarded as a classical PDE (hyperbolic, and with non-constant coefficients) – albeit not necessarily a question which would have arisen naturally in a purely classical context.*

Some of the questions which arise in this way have posed interesting challenges and have led to the import into the subject of a wide variety of techniques from the theory of linear PDE. Especially, the Princeton 1992 PhD thesis of Marek Radzikowski [4, 35, 36] (see Section 3 below) opened the way to solving some previously unsolved problems in the theory with help of techniques and results from *microlocal analysis* [5, 6, 7]. This sparked off a revolution in the subject which is still at a very active stage. See e.g. the recent papers of Fredenhagen, Koehler, Brunetti, Verch, Junker, Sahlmann and Fewster [8, 9, 10, 11, 12, 13, 14]). Radzikowski's work may be seen as having picked up where Duistermaat and Hörmander left off in 1972 when they discussed distinguished parametrices and the Feynman Propagator [6] for KG and it should be mentioned that Arthur Wightman played an important role in influencing both the earlier (cf. the last paragraph in the preface to [6]) and (as Radzikowski's PhD supervisor) the later work and in keeping the flame alive during the intervening twenty years.

The aim of the remainder of my talk will be to amplify on some of the main physical ideas and results just mentioned and to state some mathematical results related to them. In particular, I shall aim to provide some helpful physical background for workers in PDE wishing to delve into the recent literature which applies microlocal analysis.[1] I should emphasize that, aside from restricting my attention to the simple model equation, KG, I shall also mainly limit my discussion to work with which I have been personally involved. Nevertheless, I hope that what I have to say will provide a useful entrée not only to this work but also to other recent work including, in addition to the papers using microlocal analysis mentioned above, the recent mathematical work by several authors on the Hawking effect (see papers cited in Section 4) as well as the recent papers on spin-$\frac{1}{2}$ fields (which make use of Dencker's work [17] on *polarisation sets*) by Kratzert [18] and by Hollands [19], and also the papers on quantum (and the, related, classical) scattering theory by

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[1]For further general background reading on quantum field theory in curved spacetime, see the textbooks [15, 16]. Especially [16] is close in spirit to the present account.

Bachelot, Nicolas and Melnyk [20, 22, 23] for spin-$\frac{1}{2}$ fields and by Bachelot [21] for spin-1 fields on black holes.

Another purpose of this talk will be to set into context, and give a brief account of, some recent work by myself and collaborators towards a new "semi-local vacuum" concept.

In order to talk about so many things in such a short time, I shall make liberal use of short-cuts, simple examples, and analogies.

2. Particle production: the harmonic oscillator analogy

In many respects, the quantum theory of our Klein-Gordon equation, KG, in an external gravitational field is analogous to the Klein-Gordon – or Dirac – equation in an external electromagnetic field. In the latter case, if the electromagnetic field is sufficiently rapidly varying in time, it can cause the creation of electron-positron pairs out of the vacuum.
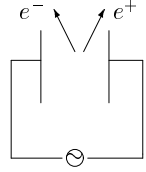

We wish to explain how this pair creation comes about by referring to an even simpler analogy: the quantum harmonic oscillator with a time-dependent frequency term. As is familiar to everyone who has taken an elementary course on Quantum Mechanics, this may be described by the time-dependent Hamiltonian:

$$\hat{H}(t) = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2(t)\hat{x}^2$$

Here $\hat{x}$ is to be thought of as analogous, in our electron-positron example, to the field strength of the quantized Dirac field, and, in the analogy with KG, to the ϕ field when it is suitably "quantized"; $\hat{p}$ is the quantum conjugate variable to $\hat{x}$, satisfying (we shall take $\hbar$ to equal 1) $[\hat{x}, \hat{p}] = i$; and the term $\omega^2(t)$ is analogous, in our electron-positron example, to the field strength of the (classical) background electromagnetic field, and in the analogy with KG, to the, generically, non-constant coefficients in KG when it is written out in some coordinate system.

Suppose, for example, that $\omega^2(t)$ takes some constant value ω_0^2 before some "initial" time T_1, returns to that same constant value after some "final" time T_2, and varies smoothly in between.
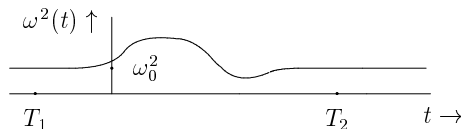
Adopting the Schrödinger representation

$$\hat{x} \mapsto x, \qquad \hat{p} \mapsto -i\frac{\partial}{\partial x}$$

$\hat{H}(t)$ maps to the differential operator

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\omega^2(t)x^2$$

(on a suitable domain in $L_{\mathbb{C}}^2(\mathbb{R}^3)$).

The eigenfunctions of the constant $\hat{H}$ for the times earlier than T_1 or later than T_2 are the usual harmonic oscillator wave functions: $\psi_0(x) = c_0 \exp(-\omega_0 x^2/2)$, $\psi_1 = c_1 H_1(\omega_0^{\frac{1}{2}} x)\exp(-\omega_0 x^2/2)$, $\psi_2 = c_2 H_2(\omega_0^{\frac{1}{2}} x)\exp(-\omega_0 x^2/2)$, ... where $c_0, c_1, c_2, \ldots$ are normalization constants and $H_0, H_1, H_2, \ldots$ the usual Hermite polynomials. For the purposes of our analogy, these should be thought of as "vacuum state", "one-particle state", "two-particle state", etc.

To understand the pair-creation phenomenon in the context of this simple model, let us take the initial vacuum state ψ_0 and evolve it according to the Schrödinger picture time-evolution for the time-dependent Hamiltonian $\hat{H}(t)$ between times T_1 and T_2. In other words, let us consider the solution to the differential equation (Schrödinger equation)

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\omega^2(t)x^2\right)\Psi(t,x) = i\frac{\partial \Psi(t,x)}{\partial t}$$

subject to the boundary condition $\Psi(T_1, x) = \psi_0(x)$.

By symmetry, the solution, $\Psi(T_2, x)$, to this problem will be an even function of x, and by completeness of the harmonic oscillator wavefunctions, it will thus have an expansion:

$$\Psi(T_2, x) = a_0 \psi_0(x) + a_2 \psi_2(x) + a_4 \psi_4(x) + \ldots$$

(with $|a_0|^2 + |a_2|^2 + |a_4|^2 + \ldots = 1$). In general, $\Psi(T_2, x)$ will not itself be a multiple of $\psi_0(x)$. In fact, as one may easily show, it will be a so-called *squeezed state* – i.e. a Gaussian $\Psi(T_2, x) = c_\alpha \exp(-\alpha x^2/2)$ with a constant, α, in the exponent which, in general will differ from ω_0. Thus the coefficients a_2, a_4, ... will not vanish. They are to be interpreted as quantum amplitudes for the creation of one, two, ... particle pairs.

Thus we have fulfilled our promise of explaining how the phenomenon of pair creation comes about. But, staying with this simple analogy, we can also get some insight into the sense (cf. Section 1) in which the concept of "particle" is vague and ambiguous: Consider the following sequence of possible modifications to the problem just discussed: First, instead of the previous $\omega^2(t)$, consider an $\omega^2(t)$ which takes on *different* constant values, say ω_{in} before T_1 and ω_{out} after T_2 and ask oneself the question: "With respect to *which* basis of harmonic oscillator wave functions should we now expand $\Psi(T_2, x)$? Those where we substitute ω_0 by ω_{in} or those where we substitute ω_0 by ω_{out}?" Next, return to the original $\omega^2(t)$ but ask oneself how one could give a particle interpretation to $\Psi(T, x)$ at a time T lying between t_1

and t_2. Finally, imagine an $\omega^2(t)$ which never settles down to any constant value, either at early or late times! As one considers these situations in turn, it gets less and less clear how, and finally impossible, to assign to any given state a definite particle-interpretation.

3. The field $*$-algebra and its states

As we mentioned in the introduction, in view of the vagueness and ambiguity of the particle concept, one wants to have a mathematical formulation which is not dependent on it. This is achieved by adopting the so-called algebraic approach to quantum field theory (see [2] and Chapter 3 in [3]). To explain how this works, we shall first say what it amounts to for the familiar example of the time-dependent harmonic oscillator just discussed:

First, one constructs the field $*$-algebra (with identity I) satisfied by the (Hermitian) *Heisenberg picture* $\hat{x}(t)$, at different times. This is determined once one specifies the commutator

$$[\hat{x}(t_1), \hat{x}(t_2)] = \hat{x}(t_1)\hat{x}(t_2) - \hat{x}(t_2)\hat{x}(t_1)$$

and one finds (e.g. one could show this with the elementary quantum mechanical formalism discussed above)

$$[\hat{x}(t_1), \hat{x}(t_2)] = i\Delta(t_1, t_2)I$$

where Δ is the difference of the classical advanced and retarded Green functions for the *classical* harmonic oscillator equation – i.e. the unique bisolution (i.e. $(d^2/dt_1^2 + \omega^2(t_1))\Delta(t_1, t_2) = 0 = (d^2/dt_2^2 + \omega^2(t_2))\Delta(t_1, t_2)$) to the *classical* harmonic oscillator equation which is antisymmetric ($\Delta(t_1, t_2) = -\Delta(t_2, t_1)$) and satisfies $\partial\Delta(t_1, t_2)/\partial t_1|_{t_2=t_1} = 1$. [This is the first manifestation of our metatheorem.]

Example: If $\omega^2(t) = \omega_0^2(= \text{constant})$, then

$$\Delta(t_1, t_2) = \frac{\sin\omega_0(t_1 - t_2)}{\omega_0}.$$

Our various Gaussian states (now thought of as Heisenberg states, unchanging in time) are now characterized by their symmetrized two-point functions

$$G_\alpha(t_1, t_2) = \langle c_\alpha \exp(-\alpha x^2/2)|(\hat{x}(t_1)\hat{x}(t_2) + \hat{x}(t_2)\hat{x}(t_1))c_\alpha \exp(-\alpha x^2/2)\rangle$$

which are to be thought of as constituting a democratic family; all values of α being on an equal footing. Again in accordance with our metatheorem, this is a family of mathematical objects which may be thought of as referring to the *classical* theory, i.e. it consists of symmetric bisolutions to the classical time-dependent harmonic oscillator equation which satisfy an additional positivity requirement (cf. Condition (c) below) – a set of objects related to the classical differential equation, this time not one which would have arisen naturally in a purely classical context.

Example: If $\omega^2(t) = \omega_0^2(= \text{constant})$ and also $\alpha = \omega_0$, then

$$G_{\omega_0}(t_1, t_2) = \frac{\cos \omega_0(t_1 - t_2)}{\omega_0}.$$

All these mathematical structures generalize to our equation, KG, provided we restrict our interest to the class of *globally hyperbolic* [24] spacetimes. We recall, [25, 26, 27] that these consist of time-orientable spacetimes which contain a Cauchy surface, and we shall assume below that a particular choice of time-orientation has been made, so we can talk about "future" and "past" in a meaningful way. For this class of spacetimes, there is a natural analogue [28, 29, 30], to the Δ we had for the harmonic oscillator, which we shall call the the *Lichnérowicz commutator function* and also denote with the symbol Δ. This is an antisymmetric distributional bisolution to KG, and, with it, one immediately obtains a natural analogue for the $*$-algebra of our harmonic oscillator example by quotienting the free $*$-algebra with identity I over $\mathbb{C}$ on abstract elements $\hat{\phi}(f), f \in C_0^\infty(\mathcal{M}; \mathbb{R})$, by the commutation relations

$$[\hat{\phi}(f_1), \hat{\phi}(f_2)] = i\Delta(f_1, f_2)I$$

$\forall f_1, f_2 \in C_0^\infty(\mathcal{M}; \mathbb{R})$, together with the relations: $\hat{\phi}(a_1 f_1 + a_2 f_2) = a_1\hat{\phi}(f_1) + a_2\hat{\phi}(f_2) \ \forall a_1, a_2 \in \mathbb{R} \ \forall f_1, f_2 \in C_0^\infty(\mathcal{M}; \mathbb{R})$ (i.e. linearity in test functions); $\hat{\phi}(f) = \hat{\phi}(f)^* \ \forall f \in C_0^\infty(\mathcal{M}; \mathbb{R})$ (i.e. Hermiticity); and $\hat{\phi}((\Box_g + m^2)f) = 0 \ \forall f \in C_0^\infty(\mathcal{M}; \mathbb{R})$ (i.e. the condition for $\hat{\phi}$ to be a weak solution of KG). The physical interpretation of the resulting $\hat{\phi}(f)$ would then be as a "smeared" quantum field "$\int_M \hat{\phi}(x)f(x)|\det(g)|^{\frac{1}{2}}d^4x$" for test-function f.

Further, one may define natural analogues for the symmetrized two point functions, G_α, of our harmonic oscillator Gaussian states to be elements[2] of the set of all bidistributions on $\mathcal{M}$ – i.e. of bilinear functionals

$$G : C_0^\infty(\mathcal{M}; \mathbb{R}) \times C_0^\infty(\mathcal{M}; \mathbb{R}) \to \mathbb{R}$$

satisfying the usual continuity properties – such that $\forall f_1, f_2, f \in C_0^\infty(\mathcal{M}; \mathbb{R})$,

(a) *(symmetry)* $G(f_1, f_2) = G(f_2, f_1)$

(b) *(distributional bi-solution property)* $G((\Box_g + m^2)f_1, f_2) = 0 = G(f_1, (\Box_g + m^2)f_2)$

(c) *(positivity)* $G(f, f) \geq 0$ *and* $G(f_1, f_1)^{\frac{1}{2}}G(f_2, f_2)^{\frac{1}{2}} \geq |\Delta(f_1, f_2)|$

(d) *(Hadamard condition)* "$G(x_1, x_2) = -\frac{1}{2\pi^2}\left(U(x_1, x_2)\mathrm{P}\frac{1}{\sigma} + V(x_1, x_2)\log|\sigma| + W(x_1, x_2)\right)$"

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[2]The set of bidistributions, G on $\mathcal{M}$, satisfying Conditions (a), (b), and (c) is in one-one correspondence with the set of so-called *quasifree states* (see e.g. [48]). This is only a small subclass of the set of all *states* (in the "algebraic sense" [2], i.e. positive, normalized, linear functionals on the field $*$-algebra). However, for every non-quasi-free state, there will be a quasifree state with the same two-point function, so the non-quasi-free states differ only in their n-point functions for $n \neq 2$; cf. [48]. Moreover the quasi-free Hadamard states, i.e. those which in addition satisfy Condition (d), play an important role in that, as conjectured in [31, 32] and proved in [33], they are *locally quasi-equivalent* and thus determine a unique local *folium* (see [2]) of states and this is believed to be the physically relevant local folium.

The last condition, the Hadamard condition, has no analogue for systems with a finite number of degrees of freedom such as our one-degree-of-freedom harmonic oscillator model and is a restriction on the nature of the (necessary) singularity in the "unsmeared" G (elsewhere locally a smooth function) for pairs of points which are null separated. It is believed to be a physically necessary condition, motivated in part by the equivalence principle, and in part by the requirement that the state denoted by the G in question should give a finite expectation value for the quantum (renormalized) stress-energy tensor (see especially [34]). A naïve statement of the condition is that the formula in quotes above should hold locally (and in the sense of smooth functions for nearby pairs of non-null-separated points) where σ, U, V, and W are all defined on a neighbourhood of the diagonal in $\mathcal{M} \times \mathcal{M}$; σ denotes the squared geodesic interval between x_1 and x_2, U (normalized so that $U(x,x) = 1$) and V are smooth two-point functions which are determined by a standard procedure due to Hadamard (U in closed form and V, by certain *Hadamard recursion relations*, as a formal series in powers of σ with coefficients which are smooth two-point functions) in terms of the geometry alone, while W (determined by another, standard, set of Hadamard recursion relations, again as a formal series in powers of σ with coefficients which are smooth functions) also depends on, and characterises, the state in question. But to spell out a mathematically meaningful statement of the condition, more must be said to replace the formal power series by genuine smooth functions, and also one apparently (but see Radzikowski's local-to-global theorem below) needs to supplement this specification of the singularity for nearby null separated points, by a statement to the effect that G is non-singular at all pairs of spacelike separated points. We refer to [3] for all the details.

What I wish to emphasize here is firstly that this condition has played a very important role in most of the deeper mathematical results concerning "quantized KG", secondly that the "mathematically messy" nature of the condition has been one of the major causes of technical difficulties in establishing these results, and thirdly that, as I have already mentioned in the introduction, an important breakthrough was achieved in 1992 when Marek Radzikowski succeeded in replacing the condition with a technically much cleaner statement [4, 35] couched in the language of microlocal analysis, namely:

(d') *(Wave Front Set [or Microlocal] Spectrum Condition)* $WF(G + i\Delta) =$

$\{(x_1, p_1; x_2, p_2) \in T^*(\mathcal{M} \times \mathcal{M} \setminus \mathbf{0}) \mid x_1$ and x_2 lie on a single null geodesic, p_1 is tangent to that null geodesic and future pointing, and p_2 when parallel transported along that null geodesic from x_2 to x_1 equals $-p_1\}$

Here, we denote elements of the cotangent bundle of $\mathcal{M}$ by pairs (x,p), x an element of $\mathcal{M}$, and p a covector at x; $\mathbf{0}$ denotes the zero section in $T^*(\mathcal{M} \times \mathcal{M})$; and we say that a covector is tangent to a curve at a point if the vector obtained by "raising an index" with the metric is tangent to the curve at that point.

Radzikowski in fact proved [4, 35] that, in the presence of Conditions (a), (b) and (c), Conditions (d) and (d') are equivalent.

Examples

Example 1: $\Box\phi = 0$ in Minkowski space $(\mathbb{R}^4, \eta)$ (i.e. KG0 with $m^2 = 0$):

$$\Delta = \delta(\sigma)\varepsilon(t_1 - t_2)$$

where $\sigma = \eta_{ab}(x_1^a - x_2^a)(x_1^b - x_2^b)$ and $\varepsilon(s)$ is the step function which is equal to 1 if $s > 0$, and to -1 if $s < 0$.

This example is of course so special that our principle of democracy doesn't hold for it: Exceptionally, there is a preferred symmetrized two point function (i.e. that of the usual *Minkowski vacuum state*) which we shall call G_0:

$$G_0 = -\frac{1}{2\pi^2}\mathrm{P}\frac{1}{\sigma}.$$

Equivalently,

$$G_0 + i\Delta = -\frac{1}{2\pi^2}\frac{1}{\sigma - 2i\epsilon(t_1 - t_2) - \epsilon^2}$$

where we use the usual informal "epsilon notation" to denote a distribution which arises as the boundary value of an analytic function.

Example 2: (Strictly, "Examples 2 and 3" are not examples but rather lower dimensional analogues. Especially, note that the Hadamard condition (d) needs to be adapted appropriately to $1 + 1$ dimensions – see [59].) KG0 with $m^2 = 0$ in $1+1$-dimensional Minkowski space $(\mathbb{R}^2, \eta)$. This is of course just the standard $1+1$-dimensional wave equation, which, with the usual double-null coordinates (U, V), may be written

$$\frac{\partial^2 \phi}{\partial U \partial V} = 0.$$

$$\Delta = \frac{1}{2}\varepsilon(U_1 - U_2)\varepsilon(V_1 - V_2).$$

Picture of Δ:

Again, there is a preferred symmetrized two point function which we shall also call G_0 and which is specified by

$$G_0 + i\Delta = -\frac{1}{2\pi}\log[(U_1 - U_2 - i\epsilon)(V_1 - V_2 - i\epsilon)]. \tag{$*$}$$

Actually, equation $(*)$ suffers from the well-known [37, 38] ill-definedness of quantized massless fields in $1+1$ dimensions, and we should really say that $G_0 + i\Delta$ only makes unambiguous mathematical sense after it has been at least once differentiated, and then that $(*)$ is just shorthand for the triplet of equations:

$$\partial^2(G_0 + i\Delta)/\partial U_1 \partial U_2 = -(1/2\pi)(U_1 - U_2 - i\epsilon)^{-2}, \quad \partial^2(G_0 + i\Delta)/\partial U_1 \partial V_2 = 0,$$

$$\partial^2(G_0 + i\Delta)/\partial V_1 \partial V_2 = -(1/2\pi)(V_1 - V_2 - i\epsilon)^{-2}. \tag{$**$}$$

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Example 3: KG with $m^2 = 0$ in a given $1 + 1$-dimensional (globally hyperbolic) curved spacetime $(\mathcal{M}, g)$:

To analyze this example, we exploit the fact that one can always (locally) find double-null coordinates, (U, V) so that $g = C(U, V)dU\,dV$ for some C^∞ function C of two variables. It is clear that these coordinates are only fixed up to reparametrizations

$$U \mapsto \hat{U}(U), \quad V \mapsto \hat{V}(V),$$

– below we shall refer to these as *Virasoro reparametrizations* because of the obvious resemblance to that concept from string theory – and that, under these, $C(U, V)$ should be transformed to

$$C(\hat{U}, \hat{V}) = \frac{dU}{d\hat{U}} \frac{dV}{d\hat{V}} C(U, V).$$

One easily sees that KG on $(\mathcal{M}, g)$ arises as the two-dimensional wave equation of Example 2 with respect to any of these choices of double-null coordinates. The appropriate choice of Δ is then clearly locally identical with that of Example 2 for any choice of (U, V) coordinates; one easily sees that it is unchanged under Virasoro reparametrizations. On the other hand, equations $(*)$ (or if you prefer $(**)$) above defining G_0 in Example 2 are *not* unchanged under Virasoro reparametrizations. Because of this, one gets lots of different symmetrized two-point functions, G, on $(\mathcal{M}, g)$ each one restricting, locally, to formula $(*)$ (or $(**)$) for a different choice of (U, V) coordinates. No one of these G is to be preferred over any other, in exemplification of our principle of democracy.

[*end of Examples*]

The first application of the equivalence between Conditions (d) and (d') was to the proof of the following theorem:

Local-to-Global Theorem (Radzikowski) (a more general result is proven in [4, 36]): *Given a globally hyperbolic spacetime $(\mathcal{M}, g)$ and letting Δ denote its Lichnérowicz commutator function for KG, then if a bidistribution G on $\mathcal{M}$ satisfies Conditions (a), (b), and (c) globally on $\mathcal{M}$ and also satisfies Condition (d) separately on each element of an open cover of $\mathcal{M}$, then it actually satisfies Condition (d) globally on all of $\mathcal{M}$.*

This theorem confirmed the correctness of a conjecture which I had made a few years earlier [32, 39]. Historically, it had been the difficulty of settling this conjecture directly which had been the stimulus for the work by Radzikowski which led to his discovery of the Wave-Front Set spectrum condition (d').

4. The Hawking effect

As a preliminary, we first discuss the *Unruh effect* which, interestingly, was discovered simultaneously, in quite different contexts and with quite different motivations, by Unruh [40] and by Bisognano and Wichmann [41]. The setting for this is quantum field theory in Minkowski space $(\mathbb{R}^4, \eta)$, and we shall illustrate it with KG0 $((\Box + m^2)\phi = 0)$. Consider the symmetrized two point function, G_0, of the

standard vacuum state, but restrict attention to the right wedge $R = \{(t, x, y, z) \in \mathbb{R}^4 \mid x > |t|\}$
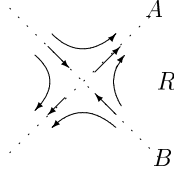
and take "time-evolution" to be the one-parameter family of wedge-preserving Lorentz boosts. Then the Unruh effect is the fact that G_0 becomes the symmetrized two point function of a thermal equilibrium state at "temperature" $1/2\pi$.

To illustrate how this comes about, take the $1 + 1$-dimensional massless case (i.e. Example 2) and focus on the future boundary of the right wedge, i.e. the right half of the null plane (now line!), A, i.e. $\{(t, x) \in \mathbb{R}^2 \mid t = x, t > 0\}$. In terms of the (U, V) coordinates of Example 2, this is $\{(U, V) \in \mathbb{R}^2 \mid U > 0, V = 0\}$ and Lorentz boosts act as dilations: $U \to e^\tau U$ (where τ is the usual *rapidity*). Defining a new coordinate u by $U = e^u$, these dilations become τ-translations and, in terms of them, one easily sees from Formula $(\ast\ast)$ in Example 2 above that

$$\frac{\partial^2 (G_0 + i\Delta)}{\partial u_1 \partial u_2} = -\frac{1}{2\pi} \left(\exp\left(\frac{u_1 - u_2}{2}\right) - \exp\left(\frac{u_2 - u_1}{2}\right) - i\epsilon \right)^{-2}$$

which is easily recognizable as the (twice differentiated) two point function of a thermal state at "temperature" $1/2\pi$ restricted to left-moving modes. (And a similar story of course holds for the past boundary of the wedge/right-movers.)

There are many mathematical results related to the Hawking effect (see e.g. [42, 43, 44, 45, 46, 47]). We briefly discuss one of them. Simplifying the wording slightly (the full statement includes some further technicalities) a special case of it is:

Theorem ([3], see also the generalization in [48]): *On the maximally extended Schwarzschild spacetime of mass M, there is a unique Schwarzschild-isometry-invariant bidistribution, G, satisfying Conditions (a), (b), (c), (d), and moreover the corresponding quantum state, when restricted to the exterior Schwarzschild region is a thermal state at the Hawking temperature $1/8\pi M$.*

We remark (cf. the classic paper of Rindler [49]) that to visualize the Kruskal spacetime one can re-interpret the picture of Minkowski space drawn above according to the substitutions:

Minkowski space $\to$ maximally extended Schwarzschild spacetime

null planes A and B $\to$ horizons

right wedge, R (each point a copy of $\mathbb{R}^2$) $\to$ exterior Schwarzschild region (each point a copy of $\mathbb{S}^2$)

one-parameter family of wedge-preserving Lorentz boosts $\to$ Schwarzschild isometries

Moreover, the above theorem also respects this *Rindler analogy* in the sense that it remains true if we substitute KG by $KG0$ and substitute the phrases to the right

of the above arrows by the phrases to the left, provided we also appropriately change the Hawking temperature from $1/8\pi M$ to $1/2\pi$.

One of the key steps in the proof of this theorem, when adapted to the latter Minkowski space version, is the demonstration that, for *any* bidistribution G satisfying the conditions of the theorem,

$$\frac{\partial^2 (G + i\Delta)}{\partial U_1 \partial U_2}\bigg|_{V_1 = V_2 = 0} = -\frac{1}{4\pi}(U_1 - U_2 - i\epsilon)^{-2}\delta(y_1 - y_2)\delta(z_1 - z_2).$$

Here we use coordinates (U, V, y, z) where $U = t + x$ and $V = t - x$, so, geometrically, the restriction is to the null plane, A, which, back in the Schwarzschild case, corresponds to a horizon.

The uniqueness part of the theorem flows from the manifest uniqueness of the right hand side of this equation. The statement about thermality flows from the observation that the right hand side of this equation is identical (except for the delta function terms) with the two point function for the Minkowski vacuum state of our Example 2, combined with the remarks we made in the second paragraph of this section in explanation of the Unruh effect for Example 2.

This demonstration was quite difficult and we refer to [3] for the details. The interested reader may find the following, much easier, exercise a useful preliminary:

Exercise: *Show that the above formula holds in the case one substitutes for $G + i\Delta$, the special value (i.e. $G_0 + i\Delta$ of Example 1 in the case $m^2 = 0$):*

$$G_0 + i\Delta = -\frac{1}{2\pi^2}[(U_1 - U_2 - i\epsilon)(V_1 - V_2 - i\epsilon) - (y_1 - y_2)^2 - (z_1 - z_2)^2]^{-2}.$$

5. Ruling out time-machines

So far, we have stayed within the realm of globally hyperbolic spacetimes. But it is also interesting to ask to what extent the theory might generalize to non-globally hyperbolic spacetimes. Starting with [53], one recently much discussed direction in which to attempt generalization is to ask about spacetimes in which a time-machine gets manufactured. (See also [50] for other non-globally hyperbolic spacetimes.) Such a spacetime must, by arguments due Hawking [51], schematically look like
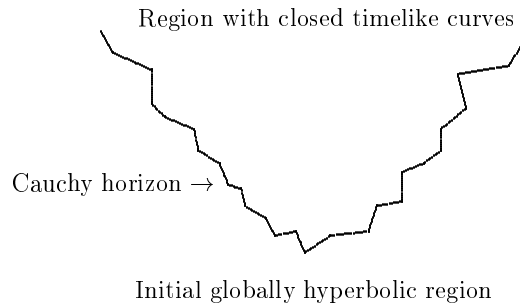
The whole spacetime will be time-orientable and contain an initial globally hyperbolic region, but there will also be a region with closed timelike curves such that the two regions share, as their common boundary, a *compactly generated* (see [51])

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Cauchy horizon. (We recall that, in general, a Cauchy horizon is necessarily a – not-necessarily smooth, see [52] – null surface.) Within the framework described in the present talk, I, Radzikowski and Wald ([54], see also [55, 56]) obtained several no-go theorems for a class of spacetimes which includes all spacetimes which tend to suggest the conclusion that (at least "semiclassically describable" – see [57]) time machines are not physically realizable. Special (actually weaker) cases of these theorems are

Theorem 1 [54]: *On such a spacetime, there does not exist any antisymmetric distributional bisolution Δ such that any neighbourhood of any point contains a globally hyperbolic subneighbourhood on which Δ coincides with the intrinsic Lichnérowicz commutator function of that subneighbourhood.*

(In the terminology of [50], Theorem 1 amounts to the statement that this class of spacetimes is *non-F-quantum compatible.*)

Theorem 2 [54]: *On such a spacetime, there does not exist any bidistribution, G, satisfying Conditions (a) and (b) and a weak local version of Condition (d).*

The proof of each of these theorems makes use of a geometrical lemma to the effect that the Cauchy horizon of such a spacetime, $(\mathcal{M}, g)$, must necessarily contain certain special points (called *base points* in [54]) with the property that every globally hyperbolic neighbourhood $\mathcal{N}$ of a base point contains two other points, r, and s, say, located inside the initial globally hyperbolic region of $(\mathcal{M}, g)$ and such that r and s are connected by a null geodesic in the total spacetime, but cannot be connected by a causal curve lying within $\mathcal{N}$. In the case of each theorem, this geometrical lemma is then easily combined with a suitable microlocal *propagation of singularities theorem* ([7] Vol. IV), and the fact that on sufficiently small neighbourhoods of a globally hyperbolic spacetime, $(\mathcal{N}, g)$, the Lichnérowicz commutator function of $(\mathcal{N}, g)$ (respectively, any bidistribution, G, on $(\mathcal{N}, g)$ satisfying Conditions (a) and (b) and the weak local version of (d)) is only singular for null separated pairs of points, to obtain a contradiction.

6. A new semi-local vacuum concept

The principle of democracy amongst our symmetrized two-point functions, G, is very fine. But, in practice, we *can* tell the difference between, say, an empty room and a room containing 10 tons of lead!

empty room   room containing 10 tons of lead
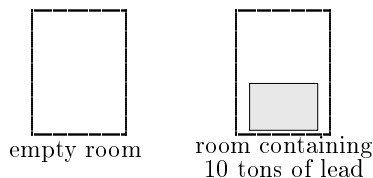
As was recognized as early as 1976 by Hajicek [58], there surely ought to be some way to reflect this in the theory of quantized KG with an appropriate "approximate local vacuum" concept. However, up to now, as far as I am aware, no clean mathematical notion for such a thing has ever been formulated.

I believe that what is lacking is something along the following lines:

Idea for a Definition: Given a, say, globally hyperbolic spacetime $(\mathcal{M}, g)$ and given any point $p \in \mathcal{M}$ then *a symmetrized two point function, G_p, for a semi-local vacuum state around p* is a bidistribution, G_p, defined on a neighbourhood $\mathcal{N}_p$ of p and satisfying conditions (a), (b), (c) (with Δ interpreted as the restriction of the Lichnérowicz commutator function from $\mathcal{M}$ to $\mathcal{N}_p$) and (d) on $\mathcal{N}_p$ which also satisfies the additional property that G_p on $\mathcal{N}_p$ looks as closely as possible near p to G_0 for $KG0$ on Minkowski space.

The problem is of course to decide on a suitable precise notion to substitute for the phrase "looks as closely as possible near p to G_0 for $KG0$ on Minkowski space".

In the case of the $1 + 1$ dimensional massless scalar field (i.e. our Example 3) I have, with help from Andrew Borrott and Claes Cramer [59], arrived at the following notion which, it will be argued in [59], gives a very satisfactory solution to this problem: One defines G_p amongst the class of G described in Example 3 by fixing the Virasoro invariance in C with the demands[3]:

$$(U, V)(p) = (0, 0), \quad C(0, 0) = 1, \quad \left.\frac{\partial^n C}{\partial U^n}\right|_{(0,0)} = 0 = \left.\frac{\partial^n C}{\partial V^n}\right|_{(0,0)} \quad \forall n \in \mathbb{N}.$$

I am presently working together with Stefan Hollands [60] on some ideas towards a suitable counterpart to the above notion in the case of massive fields and $1 + 3$ dimensions.

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[3]I call such coordinates *superspecial double-null coordinates around p*. These give a C^∞ notion of "local vacuum". One can also have a stronger notion of "local vacuum" by taking what I call *hyperspecial coordinates* i.e. by choosing $(U, V)(p) = (0, 0)$, $C(0, 0) = 1$, and demanding that, inside a suitable neighbourhood, $\mathcal{N}_p$, of p, $C(U, 0)$ and $C(0, V)$ take the value 1 throughout $\mathcal{N}_p$ – i.e., thinking geometrically, $C(r, p)$ and $C(p, s)$ take the value 1 throughout $\mathcal{N}_p$ as r and s range over each of the two null geodesics which pass through p.

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